

# ON AFFINELY CLOSED HOMOGENEOUS SPACES

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*To V. N. Latyshev on his 70-th anniversary*

**ABSTRACT.** Affinely closed homogeneous spaces  $G/H$ , i.e., affine homogeneous spaces that admit only the trivial affine embedding, are characterized for any affine algebraic group  $G$ . As a corollary, a description of affine  $G$ -algebras with finitely generated invariant subalgebras is obtained.

## 1. INTRODUCTION

Let  $G$  be an affine algebraic group over an algebraically closed field  $\mathbb{k}$  of characteristic zero and  $H$  an algebraic subgroup of  $G$ . By Chevalley's Theorem, the homogeneous space  $G/H$  admits the canonical structure of a quasiprojective variety. An *embedding* of the homogeneous space  $G/H$  is an algebraic  $G$ -variety  $X$  with a base point  $x \in X$  such that the orbit  $Gx$  is dense (and open) in  $X$  and the stabilizer  $G_x$  equals  $H$ . We denote an embedding as  $G/H \hookrightarrow X$ . We say that an embedding is *trivial* if  $Gx = X$ . An embedding  $G/H \hookrightarrow X$  is said to be *affine* if the variety  $X$  is affine. It is easy to show (for example, see [PV89, Th.1.6]) that a space  $G/H$  admits an affine embedding if and only if  $G/H$  is a quasi-affine variety, or, equivalently,  $H$  may be realized as the stabilizer of a vector in a finite-dimensional  $G$ -module. In this case the subgroup  $H$  is said to be *observable* in  $G$ . An effective description of observable subgroups in an affine algebraic group  $G$  was obtained by A. Sukhanov [Su88].

The following definition was introduced in [AT01].

**Definition 1.** A homogeneous space  $G/H$  is called *affinely closed* if it admits only the trivial affine embedding.

An affinely closed homogeneous space is automatically affine. The consideration of this class of homogeneous spaces is motivated by the

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following question: when does "the stabilizer of a point  $x$  on an affine  $G$ -variety  $X$  equals  $H$ " imply "the orbit  $Gx$  is closed" ?

For a reductive group  $G$  a homogeneous space  $G/H$  is an affine variety if and only if the subgroup  $H$  is reductive (Matsushima's Criterion). Note that for an arbitrary affine algebraic group  $G$  a description of affine homogeneous spaces  $G/H$  in group-theoretic terms for the pair  $(G, H)$  is an open problem, for more information see [Gr97, Ch.2].

For reductive  $G$  a description of affinely closed homogeneous spaces follows directly from the result due to D. Luna [Lu75]:

**Theorem 1.** *Let  $G$  be a reductive group. A homogeneous space  $G/H$  is affinely closed if and only if the subgroup  $H$  is reductive and has a finite index in its normalizer  $N_G(H)$ . Moreover, if  $G$  acts on an affine variety  $X$  and the stabilizer of a point  $x \in X$  contains a reductive subgroup  $H$  such that the group  $N_G(H)/H$  is finite, then the orbit  $Gx$  is closed.*

For example, for a maximal torus  $T$  of a reductive group  $G$  the Weyl group  $W = N_G(T)/T$  is finite, hence  $G/T$  is affinely closed. If  $\rho : H \rightarrow SL(V)$  is an irreducible representation of a semisimple group, then  $SL(V)/\rho(H)$  is affinely closed (by the Schur Lemma, the group  $N_{SL(V)}(\rho(H))/\rho(H)$  is finite). Thus the class of affinely closed homogeneous spaces is wide.

In [Ar03], affinely closed homogeneous spaces of a reductive group  $G$  play a key role in a classification of affine  $G$ -algebras such that any invariant subalgebra is finitely generated. Characterizations of complex affinely closed homogeneous spaces of reductive groups in terms of compact transformation groups and invariant algebras on compact homogeneous spaces are given in [La99] and [GL01].

The aim of this paper is to generalize the result of D. Luna to the case of an arbitrary affine algebraic group  $G$  (Theorem 2) and to obtain a classification of affine  $G$ -algebras with finitely generated invariant subalgebras (Theorem 3). Note that a characteristic-free variant of these results for a solvable group  $G$  is given in [Te04].

## 2. A DESCRIPTION OF AFFINELY CLOSED SPACES

Let us fix a Levi decomposition  $G = LG^u$  of the group  $G$  in a semidirect product of a reductive subgroup  $L$  and the unipotent radical  $G^u$ . By  $\phi$  denote the homomorphism  $\phi : G \rightarrow G/G^u$ . We shall identify the image of  $\phi$  with  $L$ . Put  $K = \phi(H)$ .

**Theorem 2.** *The following conditions are equivalent:*

- (1)  $G/H$  is affinely closed;
- (2)  $L/K$  is affinely closed.

*Proof.* The subgroup  $H$  is observable in  $G$  if and only if the subgroup  $K$  is observable in  $L$  [Su88], [Gr97, Th.7.3].

Suppose that  $L/K$  admits a non-trivial affine embedding. Then there are an  $L$ -module  $V$  and a vector  $v \in V$  such that the stabilizer  $L_v$  equals  $K$  and the orbit boundary  $Y = Z \setminus Lv$ , where  $Z = \overline{Lv}$ , is nonempty. Let  $I(Y)$  be the ideal in  $\mathbb{k}[Z]$  defining the subvariety  $Y$ . Recall that for an action of an algebraic group  $G$  on an affine variety  $X$  any element  $f \in \mathbb{k}[X]$  belongs to a finite-dimensional invariant subspace, or, equivalently,  $\mathbb{k}[X]$  is a sum of its finite-dimensional  $G$ -submodules. Thus there exists an  $L$ -submodule  $V_1 \subset I(Y)$  that generates  $I(Y)$  as an ideal. The inclusion  $V_1 \subset \mathbb{k}[Z]$  defines  $L$ -equivariant morphism  $\psi : Z \rightarrow V_1^*$  and  $\psi^{-1}(0) = Y$ . Then  $L$ -equivariant morphism  $\xi : Z \rightarrow V_2 = V_1^* \oplus (V \otimes V_1^*)$ ,  $z \rightarrow (\psi(z), z \otimes \psi(z))$  maps  $Y$  to the origin and is injective on the open orbit in  $Z$ . Hence we obtain an embedding of  $L/K$  in an  $L$ -module such that the closure of the image of this embedding contains the origin. Put  $v_2 = \xi(v)$ . By the Hilbert-Mumford Criterion, there is a one-parameter subgroup  $\lambda : \mathbb{k}^* \rightarrow L$  such that  $\lim_{t \rightarrow 0} \lambda(t)v_2 = 0$ . Consider the weight decomposition  $v_2 = v_2^{(i_1)} + \dots + v_2^{(i_s)}$  of the vector  $v_2$ , where  $\lambda(t)v_2^{(i_k)} = t^{i_k}v_2^{(i_k)}$ . Here all  $i_k$  are positive.

By the identification  $G/G^u = L$ , one may consider  $V_2$  as a  $G$ -module. Let  $W$  be a finite-dimensional  $G$ -module with a vector  $w$  whose stabilizer equals  $H$ . Replacing the pair  $(W, w)$  by the pair  $(W \oplus (W \otimes W), w + w \otimes w)$ , one may suppose that the orbit  $Gw$  intersects the line  $\mathbb{k}w$  only at the point  $w$ . For a sufficiently large  $N$  in the  $G$ -module  $W \otimes V_2^{\otimes N}$  one has  $\lim_{t \rightarrow 0} \lambda(t)(w \otimes v_2^{\otimes N}) = 0$  ( $\lambda(\mathbb{k}^*)$  may be considered as a subgroup of  $G$ ). On the other hand, the stabilizer of  $w \otimes v_2^{\otimes N}$  coincides with  $H$ . This implies that the space  $G/H$  is not affinely closed.

Conversely, suppose that  $G/H$  admits a non-trivial affine embedding. This embedding corresponds to a  $G$ -invariant subalgebra  $A \subset \mathbb{k}[G/H]$  containing a non-trivial  $G$ -invariant ideal  $I$ . Note that the algebra  $\mathbb{k}[L]$  may be identified with the subalgebra in  $\mathbb{k}[G]$  of (left- or right-)  $G^u$ -invariant functions,  $\mathbb{k}[G/H]$  is realized in  $\mathbb{k}[G]$  as the subalgebra of right  $H$ -invariants, and  $\mathbb{k}[L/K]$  is the subalgebra of left  $G^u$ -invariants in  $\mathbb{k}[G/H]$ . Consider the action of  $G^u$  on the ideal  $I$ . By the Lie-Kolchin Theorem, there is a non-zero  $G^u$ -invariant element in  $I$ . Thus the subalgebra  $A \cap \mathbb{k}[L/K]$  contains the non-trivial  $L$ -invariant ideal  $I \cap \mathbb{k}[L/K]$ . If the space  $L/K$  is affinely closed then we get a contradiction with the following lemma.

**Lemma 1.** *Let  $L/K$  be an affinely closed space of a reductive group  $L$ . Then any  $L$ -invariant subalgebra in  $\mathbb{k}[L/K]$  is finitely generated and does not contain non-trivial  $L$ -invariant ideals.*

*Proof.* Let  $B \subset \mathbb{k}[L/K]$  be a non-finitely generated invariant subalgebra. For any chain  $W_1 \subset W_2 \subset W_3 \subset \dots$  of finite-dimensional  $L$ -invariant submodules in  $\mathbb{k}[L/K]$  with  $\cup_{i=1}^{\infty} W_i = \mathbb{k}[L/K]$ , the chain

of subalgebras  $B_1 \subset B_2 \subset B_3 \subset \dots$  generated by  $W_i$  does not stabilize. Hence one may suppose that all inclusions here are strict. Let  $Z_i$  be the affine  $L$ -variety corresponding to the algebra  $B_i$ . The inclusion  $B_i \subset \mathbb{k}[L/K]$  induces the dominant morphism  $L/K \rightarrow Z_i$  and Theorem 1 implies that  $Z_i = L/K_i$ ,  $K \subset K_i$ . But  $B_1 \subset B_2 \subset B_3 \subset \dots$ , and any  $K_i$  is strictly contained in  $K_{i-1}$ , a contradiction. This shows that  $B$  is finitely generated and, as proved above,  $L$  acts transitively on the affine variety  $Z$  corresponding to  $B$ . But any non-trivial  $L$ -invariant ideal in  $B$  corresponds to a proper  $L$ -invariant subvariety in  $Z$ .  $\square$

Theorem 2 is proved.  $\square$

**Corollary 1.** *Let  $G/H$  be an affinely closed homogeneous space. Then for any affine  $G$ -variety  $X$  and a point  $x \in X$  such that  $Hx = x$ , the orbit  $Gx$  is closed.*

*Proof.* The stabilizer  $G_x$  is observable in  $G$ , hence  $\phi(G_x)$  is observable in  $L$ . The subgroup  $\phi(G_x)$  contains  $K = \phi(H)$ , and Theorems 1 and 2 imply that the space  $L/\phi(G_x)$  is affinely closed. By Theorem 2, the space  $G/G_x$  is affinely closed.  $\square$

In particular, we get

**Corollary 2.** *If  $X$  is an affine  $G$ -variety and a point  $x \in X$  is  $T$ -fixed, where  $T$  is a maximal torus of the group  $G$ , then the orbit  $Gx$  is closed.*

### 3. $G$ -ALGEBRAS WITH FINITELY GENERATED INVARIANT SUBALGEBRAS

Below an *affine algebra* over a field  $\mathbb{k}$  means a finitely generated associative commutative  $\mathbb{k}$ -algebra with unit. Let  $F$  be a subgroup of the automorphism group of an affine algebra  $\mathcal{A}$  and  $\text{rad}(\mathcal{A})$  the set of nilpotent elements of the algebra  $\mathcal{A}$ . Clearly,  $\text{rad}(\mathcal{A})$  is an  $F$ -invariant ideal in  $\mathcal{A}$ .

**Lemma 2.** *The following conditions are equivalent:*

- (1) *any  $F$ -invariant subalgebra in  $\mathcal{A}$  is finitely generated;*
- (2) *any  $F$ -invariant subalgebra in  $\mathcal{A}/\text{rad}(\mathcal{A})$  is finitely generated and  $\dim \text{rad}(\mathcal{A}) < \infty$ .*

*Proof.* Any finite-dimensional subspace in  $\text{rad}(\mathcal{A})$  generates a finite-dimensional subalgebra in  $\mathcal{A}$ . Hence if  $\dim \text{rad}(\mathcal{A}) = \infty$ , then the subalgebra generated by this subspace is not finitely generated. On the other hand, the preimage in  $\mathcal{A}$  of any non-finitely generated subalgebra in  $\mathcal{A}/\text{rad}(\mathcal{A})$  is not finitely generated.

Conversely, suppose that (2) holds. Then any subalgebra in  $\mathcal{A}$  is generated by elements whose images generate the image of this subalgebra in  $\mathcal{A}/\text{rad}(\mathcal{A})$ , and by a basis of the radical of the subalgebra.  $\square$

By definition, a  $G$ -algebra is an affine algebra  $\mathcal{A}$  with an action (by automorphisms) of an algebraic group  $G$  such that any element  $a \in \mathcal{A}$  is contained in a finite-dimensional  $G$ -invariant subspace, where  $G$  acts rationally. Our aim is to describe all  $G$ -algebras such that any  $G$ -invariant subalgebra is finitely generated. By Lemma 2, we may assume that  $\text{rad}(\mathcal{A}) = 0$ .

Let  $X = \text{Spec}(\mathcal{A})$  be the affine variety corresponding to affine algebra  $\mathcal{A}$  without nilpotents. To fix a structure of  $G$ -algebra on  $\mathcal{A} = \mathbb{k}[X]$  is nothing else but to fix an (algebraic)  $G$ -action on  $X$ . Define the *dimension*  $\dim \mathcal{A}$  of the algebra  $\mathcal{A}$  as the dimension of the variety  $X$ .

**Lemma 3.** *If  $\dim \mathcal{A} \leq 1$ , then any subalgebra in  $\mathcal{A}$  is finitely generated.*

*Proof.* The case when  $X$  is irreducible is considered in [Ar03, Prop. 2]. If  $X = X_1 \cup \dots \cup X_m$  is the decomposition on irreducible components, then  $\mathcal{A}$  is embedded into the direct sum of the algebras  $\mathbb{k}[X_i]$ , and any subalgebra in a summand is finitely generated. Now it is easy to finish the proof by induction on  $m$  considering the projection of  $\mathcal{A}$  on  $\mathbb{k}[X_1]$ .  $\square$

**Lemma 4.** *Suppose that  $X = Z_1 \cup Z_2$ , where  $Z_1$  and  $Z_2$  are closed invariant subvarieties. Then the following conditions are equivalent:*

- (1) *any invariant subalgebra in  $\mathcal{A}$  is finitely generated;*
- (2) *any invariant subalgebra in  $\mathbb{k}[Z_1]$  and in  $\mathbb{k}[Z_2]$  is finitely generated.*

*Proof.* If there is a non-finitely generated subalgebra in  $\mathbb{k}[Z_1]$ , then one may consider its preimage with respect to the restriction homomorphism  $\mathbb{k}[X] \rightarrow \mathbb{k}[Z_1]$ . To prove the converse, embed  $\mathbb{k}[X]$  in  $\mathbb{k}[Z_1] \oplus \mathbb{k}[Z_2]$  and use the arguments from the proof of the previous lemma.  $\square$

By Lemma 4, one may assume that  $G$  acts transitively on the set of irreducible components of the variety  $X$ .

Below we generalize a construction from [Ar03] to the case of non-connected groups and reducible varieties. Let  $Y$  be a closed subvariety of  $X$ . Consider a subalgebra

$$\mathcal{A}(X, Y) = \{f \in \mathbb{k}[X] \mid f(y_1) = f(y_2) \ \forall y_1, y_2 \in Y\}.$$

**Lemma 5.** *If  $Y$  contains an irreducible component of positive dimension that does not coincide with any irreducible component of  $X$ , then  $\mathcal{A}(X, Y)$  is not finitely generated.*

*Proof.* Note that  $\mathcal{A}(X, Y) = \mathbb{k} \oplus I(Y)$ . If  $\mathcal{A}(X, Y)$  is finitely generated, then one may assume that generators  $f_1, \dots, f_k$  are in  $I(Y)$ . Any monom in  $f_1, \dots, f_k$  of degree  $s$  is in  $I(Y)^s$ . Hence it is sufficient to prove that for some  $l$  the space  $I(Y)/I(Y)^l$  is infinite-dimensional.

Let  $Y = Y_1 \cup \dots \cup Y_n$  and  $X = X_1 \cup \dots \cup X_m$  be the decompositions on irreducible components, and  $Y_1 \subset X_1$ ,  $Y_1 \neq X_1$ ,  $\dim Y_1 > 0$ . Suppose that  $f \in I(Y)$  and  $f$  is not identically zero on  $X_1$ . Let  $\mathcal{O}_{X_1, Y_1}$  be

the local ring of the subvariety  $Y_1$  in  $X_1$  and  $\mathcal{I}$  its maximal ideal. By the Nakayama Lemma,  $\cap_{i=1}^{\infty} \mathcal{I}^i = 0$ , hence after restriction to  $X_1$  the element  $f$  belongs to  $\mathcal{I}^{l-1} \setminus \mathcal{I}^l$  for some  $l \geq 2$ . Let  $W$  be a subspace in  $\mathbb{k}[X]$  complementary to  $I(Y_1)$ . Note that  $\dim Y_1 > 0$  implies  $\dim W = \infty$ . The subspace  $fW$  may be considered as an infinite-dimensional subspace in  $\mathcal{I}^{l-1}$  with zero intersection with  $\mathcal{I}^l$ . Hence  $fW$  determines an infinite-dimensional subspace in  $I(Y)/I(Y)^l$ .  $\square$

We conclude that any invariant subvariety  $Y$  satisfying the conditions of Lemma 5, determines the non-finitely generated invariant subalgebra  $\mathcal{A}(X, Y)$  in  $\mathbb{k}[X]$ .

By  $G^0$  denote the connected component of unit of a group  $G$ .

**Theorem 3.** *Let  $\mathcal{A}$  be a  $G$ -algebra without nilpotents with the non-trivial induced action of the subgroup  $G^u$ . The following conditions are equivalent:*

- (1) *any  $G$ -invariant subalgebra in  $\mathcal{A}$  is finitely generated;*
- (2) *any  $G$ -invariant subalgebra in  $\mathcal{A}$  does not contain non-trivial  $G$ -invariant ideals;*
- (3) *any  $L$ -invariant subalgebra in  $\mathcal{A}^{G^u}$  does not contain non-trivial  $L$ -invariant ideals;*
- (4)  *$\mathcal{A} = \mathbb{k}[G/H]$ , where  $G/H$  is an affinely closed homogeneous space;*
- (5)  *$\mathcal{A}^{G^u} = \mathbb{k}[L/K]$ , where  $L/K$  is an affinely closed homogeneous space.*

*Proof.* (1)  $\Rightarrow$  (4) *Step 1.* Suppose that the action  $G : X$  is not transitive. The closure  $Y$  of a  $G$ -orbit on  $X$  is an invariant subvariety and we may apply Lemma 5 with the only exceptions  $Y = X$  and  $\dim Y = 0$ . Hence (1) implies that  $G^0$  acts on any component  $X_i$  either with an open orbit and the boundary of this orbit is a finite set of points, or trivially. In these cases the action  $G^u : X$  is trivial [Po75, Th.3], see also [Ar03, Prop.4].

*Step 2.* Suppose that the action  $G : X$  is transitive. Then  $X = G/H$ . If  $G/H$  admits a non-trivial affine embedding  $G/H \hookrightarrow X'$ , then  $\mathbb{k}[X']$  is an invariant subalgebra in  $\mathcal{A}$ . By Step 1, this subalgebra contains a non-finitely generated invariant subalgebra.

(2)  $\Rightarrow$  (4) The absence of non-trivial invariant ideals in  $\mathcal{A}$  implies that  $X = G/H$ , and the absence of non-trivial invariant ideals in invariant subalgebras implies that  $G/H$  does not admit non-trivial embeddings.

Proofs (4)  $\Rightarrow$  (1) and (4)  $\Rightarrow$  (2) are analogous to the proof of Lemma 1 (one should use Corollary 1). By the same arguments we get (5)  $\Rightarrow$  (3).

We know that  $\mathbb{k}[G/H]^{G^u} = \mathbb{k}[L/K]$ . Theorem 2 implies (4)  $\Rightarrow$  (5).

(3)  $\Rightarrow$  (2) Let  $\mathcal{B}$  be an invariant subalgebra in  $\mathcal{A}$  and  $I$  a non-trivial invariant ideal in  $\mathcal{B}$ . Then  $I \cap \mathcal{A}^{G^u}$  is a non-trivial invariant ideal in  $\mathcal{B} \cap \mathcal{A}^{G^u}$ . This completes the proof of Theorem 3.  $\square$

*Remark 1.* 1) The conditions of Theorem 3 are not equivalent to the condition "any  $L$ -invariant subalgebra in  $\mathcal{A}^{G^u}$  is finitely generated": one may consider  $G = G^u = (\mathbb{k}, +)$  acting on  $\mathbb{k}[x, y]$  by the formula  $(a, f(x, y)) \rightarrow f(x + ay, y)$ .

2) The implication (1)  $\Rightarrow$  (5) is incorrect if  $\mathcal{A} = \mathcal{A}^{G^u}$ , see Lemma 3 and [Ar03].

3) The restriction  $\mathcal{A} \neq \mathcal{A}^{G^u}$  is natural because the case of reductive group actions was studied in [Ar03] under the assumptions that  $G$  is connected and  $X$  is irreducible. But Lemma 5 and above arguments show that if a reductive  $G$  acts transitively on the set of irreducible components of  $X$ , then any invariant subalgebra in  $\mathbb{k}[X]$  is finitely generated if and only if either the  $G^0$ -algebras  $\mathbb{k}[X_i]$  for any irreducible component  $X_i$  have (in the terminology of [Ar03]) type C or HV, or  $X = G/H$  and  $G/H$  is affinely closed (in this case, the  $G^0$ -algebras  $\mathbb{k}[X_i]$  may not have type N, see example 5) below). Surprisingly, the restriction  $\mathcal{A} \neq \mathcal{A}^{G^u}$  simplifies the main results.

**Corollary 3.** *Let  $G/H$  be a quasi-affine homogeneous space. Suppose that  $H$  does not contain  $G^u$ . Then either  $G/H$  is affinely closed or there are infinitely many pairwise nonisomorphic affine embeddings  $G/H \hookrightarrow X_i$  and a sequence of dominant equivariant morphisms:*

$$X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \dots$$

*Proof.* Let  $G/H \hookrightarrow X$  be a non-trivial affine embedding and  $Y$  the complement to the open orbit in  $X$ . The algebra  $\mathcal{A}(X, Y)$  is not finitely generated. Let  $p_0 \in I(Y)$  be a non-zero element and  $f_1, \dots, f_s$  a set of generators of  $\mathbb{k}[X]$ . Put  $p_i = p_0 f_i$ ,  $i = 1, \dots, s$  and extend the set  $p_0, p_1, \dots, p_s$  to an (infinite) generating set  $p_0, p_1, \dots, p_s, h_1, h_2, \dots$  of the algebra  $\mathcal{A}(X, Y)$ . The affine  $G$ -varieties  $X_i$  corresponding to the algebras

$$\mathcal{B}_i = \mathbb{k}[\langle Gp_0, Gp_1, \dots, Gp_s, Gh_1, \dots, Gh_i \rangle] \subset \mathcal{A}(X, Y) \subset \mathbb{k}[X] \subset \mathbb{k}[G/H],$$

define embeddings  $G/H \hookrightarrow X_i$  and the inclusions of algebras determine the desired chain of dominant morphisms. In the sequence  $X_i$  there is a subsequence consisting of pairwise nonisomorphic embeddings. (By definition, an isomorphism of two embeddings sends the base point to the base point and is the unique equivariant morphism identical on the open orbit.)  $\square$

#### 4. SOME REMARKS ON AFFINE EMBEDDINGS OF HOMOGENEOUS SPACES FOR NON-REDUCTIVE GROUPS

In this section we consider elementary examples that provide negative answers to some natural questions.

Let  $V$  be a finite-dimensional  $G$ -module and  $v \in V$ .

1) *The orbit  $Gv$  is closed, but the orbit  $Lv$  is not closed.* Consider

$$V = \mathbb{k}^2, \quad v = (1, 1), \quad G = \left\{ \begin{pmatrix} t & a \\ 0 & 1 \end{pmatrix} \right\}, \quad L = \left\{ \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

2) *The orbit  $Lv$  is closed, but the orbit  $Gv$  is not closed.* Consider

$$V = \mathbb{k}^2, \quad v = (1, 1), \quad G = \left\{ \begin{pmatrix} t & a \\ 0 & t^{-1} \end{pmatrix} \right\}, \quad L = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right\}.$$

3) *The space  $G/H$  is affinely closed does not imply that all  $L$ -orbits on  $G/H$  are closed.* Consider

$$G = \left\{ \begin{pmatrix} t & a \\ 0 & t^{-1} \end{pmatrix} \right\}, \quad H = L = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right\}, \quad x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ \Rightarrow \overline{LxH} \neq LxH.$$

4) *If  $G^0/H^0$  is affinely closed then  $G/H$  is affinely closed, but the converse is not true.* One may take  $G = SL(2)$  with any finite non-Abelian subgroup  $H$ .

5) *If  $G^0/(H \cap G^0)$  is affinely closed then  $G/H$  is affinely closed, but the converse is not true.* Consider

$$G = N_{SL(2)}T, \quad H = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle.$$

6) *The condition  $0 \in \overline{Gv}$  does not imply that 0 may be obtained as the limit of a one-parameter subgroup in  $G$ .* The corresponding example for a solvable  $G$  is given in [Bi71, sec.11].

An important open problem is to characterize affinely closed spaces (for both reductive and non-reductive groups) over an algebraically closed field of positive characteristic. In particular, it is not known do we have here Corollary 1. Some results in this direction may be found in [Ar03, sec.8].

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